Costello. SUSY field theories for mathematicians

Course. 2016 Perimeter

Field theory vs String theory

$\mathbb{C}^r \to E \to X$ $\qquad$ $X \xrightarrow{u} M$

$D_A \in \mathcal{A}(E)$

eg YM, CS, BF

$\mathcal{A}(E)/\mathcal{G}(E) \sim X \to BG$

(i) h.e.

(ii) 'derived' $\qquad$ $\shortparallel$ $*/G$

Content:

- Spinors and SUSY algebras.
- SUSY Gauge Theory
- Witten's twist
- SUSY index
- Chern-Simons theory
- 7d Gauge theory, K_{DT}.

§ Spinors and SUSY algebras.

$V = \mathbb{C}^n, \langle\ \rangle$ inner product space

$\left\{ \begin{smallmatrix}(\text{infinitesimal})\\ \text{isometries}\end{smallmatrix} \right\} = \underline{so}(n,\mathbb{C}) \ltimes V$

translation (trivial Lie alg).

SUSY alg. $= \underline{so}(n,\mathbb{C}) \ltimes \underbrace{(V + \Pi S)}_{T}$ super-translat$^{\underline{n}}$

$\rightsquigarrow$ super Lie alg. $T \triangleq V \oplus \Pi S$

w/ $Spin(n,\mathbb{C}) \curvearrowright S$ cx. spinor rep.

need $\Gamma : \text{Sym}^2 S \longrightarrow V$

$[(v_1,\psi_1),(v_2,\psi_2)] = (\Gamma(\psi_1 \otimes \psi_2),\ 0)$

To constr. $Spin(n) \subset Cl_n^+ \curvearrowright \$$

e.g. $Spin(3) = S^3 \subset \mathbb{H} \curvearrowright \mathbb{H}$

unit quatenions

Clifford alg. $Cl(V,g) = \bigotimes V / u \otimes v + v \otimes u = g(u,v)$

Explicitly, V w/ orthonormal base e_i's.

$Cl_n := \mathbb{C}\langle e_1,\dots,e_n\rangle / e_i \cdot e_j + e_j \cdot e_i = \delta_{ij}$

- assoc. alg, $\simeq \Lambda^\bullet V$ as vector spaces.
- $\underline{so}(n,\mathbb{C}) \subset Cl_n$

  $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = E_{ij} - E_{ji} \longmapsto e_i \cdot e_j$

Fact: $Spin(n,\mathbb{C}) \triangleq \exp(\underline{so}(n,\mathbb{C})) \subset Cl_n$

Repr. of $Cl_{n=2m} \ni f_j^{\pm} := e_{2j-1} \pm i\, e_{2j}$ base

$\left([f_i^{\pm}, f_j^{\pm}] = 0, \quad [f_i^{+}, f_j^{-}] = 2\delta_{ij} \right)$

$\downarrow$

$S_{cl} := \mathbb{C}[f_j^-]$

f_j^- act by $f_j^- \cdot$

f_j^+ act by $2 \frac{\partial}{\partial f_j^-}$

- $\exists$'!' irred. rep. of Cl_n
- $Cl_n = \mathrm{End}(S_{cl})$
- $S_{cl} = S^+ \oplus S^-$ preserved by $\mathrm{Spin}(n, \mathbb{C}) \subseteq Cl_n$.

  $S^+ \not\cong S^-$ $\quad (\dim = 2^{m-1})$

Repr. of $Cl_{2m+1} \ni f_j^{\pm}, \; e_{2m+1}$ base

$\downarrow$

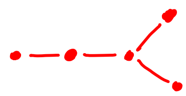
$S_{cl} := \mathbb{C}[f_j^-, \varepsilon]$, w/ e_{2m+1} act by $\varepsilon + \frac{\partial}{\partial \varepsilon}$

$S_{cl} = S^+ \oplus S^-$ preserved by $\mathrm{Spin}(2m+1, \mathbb{C})$

But $S^+ \xrightarrow[\gamma \cdot]{\sim} S^-$ w/ $\gamma := e_1 e_2 \cdots e_{2m+1} \in Cl_{2m+1}$

$\gamma^2 = \pm 1$. (indep. of choice of base)

(Call S, dim 2^m).

$SO(2n+1)$  $\quad SO(2n)$ 

$\widetilde{SO}(2)$	$\widetilde{SO}(3)$	$\widetilde{SO}(4)$	$\widetilde{SO}(5)$	$\widetilde{SO}(6)$
$= \mathbb{C}^\times$	$= SL(2)$	$= (SL(2))^2$	$= C_2 = Sp(4)$	$= A_3 = SL(4)$

$n = 2m$: $\quad n \equiv 4 \qquad n \equiv 2 \qquad n \equiv 6 \pmod 8$

$\qquad V \subseteq S_+ \otimes S_- \qquad V \subseteq S^2 S_\pm \qquad V \subseteq \Lambda^2 S_\pm$

$n = 2m+1$: $\quad n \equiv 3 \qquad n \equiv 5 \pmod 8$

$\qquad V \subseteq S^2 S \qquad V \subseteq \Lambda^2 S$

Need $\Gamma : \mathrm{Sym}^2(\underset{S_\pm}{S} \otimes \mathbb{C}^m) \longrightarrow V$!

[n=2] $\mathrm{Spin}(2,\mathbb{C}) \xrightarrow{2:1} SO(2,\mathbb{C})$ is $\mathbb{C}^\times \xrightarrow{z^2} \mathbb{C}^\times$

$$V \cong \mathbb{C}^2 = V^{1,0} \oplus V^{0,1}$$

SO(2,ℂ)-wt: 1   −1

span by ∂_z $\partial_{\bar z}$

$S_\pm \simeq \mathbb{C}$ SO(2,ℂ)-wt. $= \pm\frac{1}{2}$

$\Rightarrow$ $S^2 S^+ = V^{1,0}$ & $S^2 S^- = V^{0,1}$

$\rightsquigarrow$ (n,m)-SUSY alg.

$$S_{n,m} := S^+ \otimes \mathbb{C}^n + S^- \otimes \mathbb{C}^m$$

(assume $\mathbb{C}^n, \mathbb{C}^m$ w/ inner product $\langle\, ,\rangle$)

$\Rightarrow \Gamma : \mathrm{Sym}^2 S_{n,m} \longrightarrow V$

R-symmetry $SO(n)\times SO(m)$

[n=3] $\mathrm{Spin}(3,\mathbb{C}) \simeq SL(2,\mathbb{C}) \curvearrowright \mathbb{C}^2 = S$

$$V \simeq \mathrm{Sym}^2 S$$

$\rightsquigarrow$ $N=n$ SUSY $V \oplus \Pi(S \otimes \underset{\langle\, ,\rangle}{\mathbb{C}^n})$ w/ R-symm. $SO(n,\mathbb{C})$

[n=4] $\mathrm{Spin}(4,\mathbb{C}) = SL(2,\mathbb{C})_+ \times SL(2,\mathbb{C})_- \curvearrowright S_+ \oplus S_-$

$$V \simeq S_+ \otimes S_-$$

$\rightsquigarrow$ Γ on $S_W := S_+ \otimes W + S_- \otimes W^*$

for any $W \simeq \mathbb{C}^n$

$\rightsquigarrow$ $V \oplus \Pi S_W$ w/ R-symmetry $GL(n,\mathbb{C})$.

[n=5] $\mathrm{Spin}(5,\mathbb{C}) = Sp(4,\mathbb{C}) \curvearrowright \mathbb{C}^4 = S, \omega$ ← sympl. form.

$\mathbb{C}^5 \simeq \overset{\bullet \Rightarrow \bullet}{V \quad S}$ $\equiv$ $(U \simeq \mathbb{C}^4, \omega)$ $\overset{\bullet \Leftarrow \bullet}{\Lambda^2_0 U = \Lambda^2 U / \mathbb{C}\omega}$

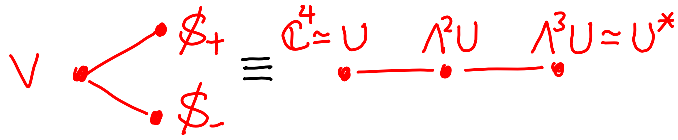
$$V = (\Lambda^2 S)/\mathbb{C}\omega \subset \Lambda^2 S,$$

$S \otimes \mathbb{C}^{2n}$, n extended SUSY, $\mathbb{C}^{2n}$ is symplectic

$\Rightarrow \exists\ \Gamma : \mathrm{Sym}^2(S \otimes \mathbb{C}^{2n}) \longrightarrow V$

R-symmetry is $Sp(2n, \mathbb{C})$.

[$n=6$]



$Spin(6, \mathbb{C}) = SL(4, \mathbb{C}) \curvearrowright \mathbb{C}^4 = S_+ = S_-^*$

$V = \Lambda^2 S_+$
$\ \ = \Lambda^2 S_-$

(n, m) extended SUSY

$S_+ \otimes \mathbb{C}^{2n} + S_- \otimes \mathbb{C}^{2m}$

R-symmetry $Sp(2n, \mathbb{C}) \times Sp(2m, \mathbb{C})$

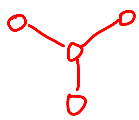
$(1,0)$ SUSY $\quad S_+ \otimes \mathbb{C}^2 \quad$ 8 supercharges

Example: M5 brane in 11d has $(2,0)$ SUSY,
Rotation of normal directions in R-symmetry
Expect $Spin(5, \mathbb{C}) = Sp(4, \mathbb{C})$ R-symmetry.

[$d=7$] S, $V \subset \Lambda^2 S$, $\dim S = 8$

n SUSY $\quad S \otimes \mathbb{C}^{2n}$, R-symmetry is $Sp(2n, \mathbb{C})$

[$d=8$] S_+, S_- 8 dimensional



$V \subseteq S_+ \otimes S_-$

n extended SUSY $\quad S_+ \otimes \mathbb{C}^n + S_- \otimes \mathbb{C}^{n*}$

R-symmetry is $GL(n, \mathbb{C})$.

[$d=10$] S_+, S_- dim 16. $\quad V \subset S^2 S_\pm$ (similar to case $d=2 = 10-8$)

(n, m) SUSY $\quad S_+ \otimes \mathbb{C}^n + S_- \otimes \mathbb{C}^m$

R-symmetry is $SO(n) \times SO(m)$.

§ SUSY Gauge Theory

<u>Claim</u>: In dim $d = 3, 4, 6, 10$ $(= 2 + \dim \mathbb{A})$

$\Rightarrow \exists$ SUSY gauge theory w/ minimal $N = \overset{1 \text{ or}}{(1,0)}$ SUSY

w/ fields: $A \in \mathcal{A}(E) = \Omega^1(\mathbb{R}^d, \mathfrak{g})$ (connection) & $\psi \in \Gamma(\mathbb{R}^d, S \otimes \mathfrak{g})$ (spinor)

d	3	4	6	10
#SUSY	S	$S_+ + S_-$	$S_+ + S_-$	S_+
	2 dim	4 dim	8 dim	16 dim

Action $S : \mathcal{A}(E) \times \Gamma(\mathbb{R}^d, S \otimes \mathfrak{g}) \longrightarrow \mathbb{C}$

$$S(A, \psi) = \int |F(A)|^2 + \langle \psi, \not{D}_A \psi \rangle$$

w/ $\not{D}_A = Cl \circ \nabla : C^\infty(M, S_\pm) \to C^\infty(M, S_\mp)$ Dirac operator

<u>Action of SUSY alg.</u> In flat space,

$Q \in S \rightsquigarrow$ symmetry of space of fields by

$$(A, \psi) \longrightarrow (A + \varepsilon \Gamma(Q, \psi), \psi + \varepsilon F(A) \cdot Q)$$

(↳ Clifford multi (=rotation).)

$\Gamma(Q, \psi) \in C^\infty(\mathbb{R}^d, T\mathbb{R}^d \otimes \mathfrak{g}) \simeq \Omega^1(\mathbb{R}^d) \otimes \mathfrak{g}$

$F(A) \in \Omega^2(\mathbb{R}^d) \otimes \mathfrak{g}$

$\Lambda^2 \mathbb{R}^d \subseteq Cl(\mathbb{R}^d)$ a copy of $\mathfrak{so}(d, \mathbb{C})$.

i.e. vector field on $\mathcal{A}(E) \times \Gamma(\mathbb{R}^d, S \otimes \mathfrak{g})$,

$$(A, \psi) \longmapsto (\Gamma(Q, \psi), F(A) \cdot Q)$$

Claim: 1) This preserves action S

2) $V_Q :=$ assoc. v.f. on space of fields, then

$$[V_Q, V_{Q'}] = \mathcal{L}_{\Gamma(Q \otimes Q')} \text{ modulo gauge \& EOM}$$

(EOM = Euler-Lagrange eqt. for S)

Main interest: Partition function

$$Z \text{ "=" } \int \mathcal{D}A \, \mathcal{D}\psi \; e^{-S(A, \psi)}$$

- $\mathcal{N}=1$, SUSY YM

$M = \mathbb{R}^{d-1,1}$ $\quad d = 2 + \dim_{\mathbb{R}} \mathbb{A}$ $\quad \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ normed alg.

Fields $(A, \psi) \in \Omega^1(M, \mathfrak{g}) \oplus \Omega^0(M, \$^+ \otimes \mathfrak{g})$

Action $S(A, \psi) = -\frac{1}{4} \int_M \langle F_A, F_A \rangle + \frac{1}{2} \int_M \langle \psi, \not{D}_A \psi \rangle$

$Q \in \$^+ \rightsquigarrow$ SUSY transform

$$\delta_Q(A, \psi) = (\Gamma(Q, \psi), \tfrac{1}{2} \varepsilon F_A \cdot \psi)$$

Goal: Prove S is SUSY-inv.

i.e. $\delta_Q \left(-\frac{1}{4} \langle F_A, F_A \rangle + \frac{1}{2} \langle \psi, \not{D}_A \psi \rangle \right) \equiv 0$ mod divergent terms.

Background. $\mathbb{R}^{d-1,1} = \left\{ \begin{pmatrix} t+x & y \\ \bar{y} & t-x \end{pmatrix} : \begin{array}{l} t, x \in \mathbb{R} \\ y \in \mathbb{A} \end{array} \right\}$

$|A| = -\det A$ $\quad$ pos. def.: x, y $\quad$ neg. def.: t

$\$^{\pm} = \mathbb{A}^2$

Clifford multi. = matrix multi. $\quad \mathbb{R}^{d-1,1} \times \underbrace{\$^+}_{\mathbb{A}^2} \longrightarrow \underbrace{\$^-}_{\mathbb{A}^2}$

Charge conjugation

$\langle \ , \ \rangle : \$^+ \times \$^- \longrightarrow \mathbb{R}$ $\quad \langle \phi, \psi \rangle = \mathrm{Re}\, \phi^* \psi$

$\Gamma : \$^+ \times \$^+ \longrightarrow \mathbb{R}^{d-1,1}$ $\quad \Gamma(\phi, \psi) = -(\phi \psi^* + \psi \phi^*)^{\sim}$

$\sim : \mathbb{R}^{d-1,1} \circlearrowleft$ $\quad t \mapsto t,\ x \mapsto -x,\ y \mapsto -y$

$$\widetilde{\begin{pmatrix} t-x & y \\ \bar{y} & t+x \end{pmatrix}} = \begin{pmatrix} t+x & -y \\ -\bar{y} & t-x \end{pmatrix}$$
$\leftarrow$ is the adjoint of a 2×2 matrix.

$$\delta_Q \langle F_A, F_A \rangle = 2 \langle F_A, d_A \Gamma(Q, \psi) \rangle \equiv 2 \langle d_A^* F_A, \Gamma(Q, \psi) \rangle$$
$$= 2 \langle \psi, (d_A^* F_A) \cdot Q \rangle = 2 \langle \psi, ((\cancel{d_A} + d_A^*) F_A) \cdot Q \rangle$$
$$= 2 \cancel{\langle \psi, \not{D}_A (F_A \cdot Q) \rangle} \quad (\because \not{D}_A Q = 0, \text{ i.e. } Q = \text{const.})$$

$$\delta_Q \langle \psi, \not{D}_A \psi \rangle = 2 \langle \psi, \not{D}_A (\delta_Q \psi) \rangle + 2 \langle \psi, (\delta_Q A) \cdot \psi \rangle$$
$$= \cancel{\langle \psi, \not{D}_A (F_A \cdot Q) \rangle} + 2 \underbrace{\langle \psi, \Gamma(Q, \psi) \cdot \psi \rangle}_{\langle Q, \Gamma(\psi, \psi) \cdot \psi \rangle}$$

Key lemma (3 ψ's rule). $\Gamma(\psi, \psi) \cdot \psi = 0$

Pf. $-2 (\widetilde{\psi \psi^*}) \cdot \psi = 0$ $\quad \left((v),\ (v v^T)^{adj} \cdot v = 0 \right)$

$= -2 \begin{pmatrix} a \\ b \end{pmatrix} \widetilde{(\bar{a}, \bar{b})} \begin{pmatrix} a \\ b \end{pmatrix} = 0$. $\quad (\because A^{adj} \cdot A = (\det A) \cdot I_{n \times n})$.

- For general manifold M,

$C^\infty(M,S) \supseteq$ cov. constant spinors
$C^\infty(M,T_M) \supseteq$ cov. constant vectors
$\}\rightsquigarrow$ smaller SUSY alg.

Eg. $d=4$, $\mathfrak{g}=\mathbb{R}$ Abelian ($\Rightarrow F_A = dA$)

Given $Q=(Q_+,Q_-)\in S_+\oplus S_-$

$$\rightsquigarrow \quad \Gamma(Q_-,\psi_+)+\Gamma(Q_+,\psi_-) \longleftrightarrow (\psi_+,\psi_-)$$

$$\Omega^1(\mathbb{R}^4) \leftrightarrows C^\infty(M,S_+\oplus S_-)$$

$$A \longmapsto (dA\cdot Q_+, dA\cdot Q_-)$$

Compute $[Q_+,Q_-]$ acts on A

$$Q_-Q_+ : A \longmapsto dA\cdot Q_+ \longmapsto \Gamma(Q_-\otimes(dA\cdot Q_+))$$

$$Q_+Q_- : A \longmapsto dA\cdot Q_- \longmapsto \Gamma(Q_+\otimes(dA\cdot Q_-))$$

Note, $dA(\Gamma(Q_+\otimes Q_-)) = \Gamma((dA\,Q_+)\otimes Q_-)+\Gamma(Q_+\otimes dA\cdot Q_-)$

$$[Q_+,Q_-] = dA\cdot\Gamma(Q_+\otimes Q_-) = \Gamma(Q_+\otimes Q_-)\vee dA$$

$$= \mathcal{L}_{\Gamma(Q_+\otimes Q_-)}A + d(\Gamma(Q_+\otimes Q_-)\vee A)$$

This is a gauge transformation

- Dimension reduction gives SUSY gauge theory, but with more fields:

$$d=10,\ \mathcal{N}=(1,0) \xrightarrow[\text{reduct}^{2}]{\text{dim.}} d=4,\ \mathcal{N}=4$$

w/ 6 scalar fields

Bosonic fields: $A_{10d} = A_{4d} + \sum_{i=1}^{6} dx_{4+i}\cdot\varphi_i$

$A_{4d}\in\Omega^1(\mathbb{R}^4)$, $\varphi_1,\dots,\varphi_6\in C^\infty(\mathbb{R}^4)$

Fermionic fields: $\psi\in S_+^{10d} = S_+^{4d}\otimes S_+^{6d} + S_-^{4d}\otimes S_-^{6d}$

$$= S_+^{4d}\otimes W + S_-^{4d}\otimes W^*$$

W = fund. rep. of $SL(4,\mathbb{C}) = Spin(6,\mathbb{C})$.

$\Rightarrow \mathcal{N}=4$ in $4d$

§ Witten's twist

Choose $\rho : \mathrm{Spin}(d) \longrightarrow G_R$

$$\mathrm{Spin}(d) \xrightarrow{(1,\rho)} \mathrm{Spin}(d) \times G_R \curvearrowright V \oplus \Pi(\$ \otimes \mathbb{C}^m)$$

Find $Q \in \$ \otimes \mathbb{C}^m$ Spin(d)-invariant

s.t. $[Q, Q] = 0$

$\Rightarrow$ SUSY admits odd symmetry Q w/ $Q^2 = 0$
($\rightsquigarrow$ take Q-cohomology)
Add Q to act on
observables/operators/Hilbert space ......
$\rightsquigarrow$ twisted theory.

Eg $d = 4$, $N = 2$ $\quad \dim W = 2$
SUSY alg. : $\mathbb{C}^4 + \Pi(S_+ \otimes W + S_- \otimes W^*)$

$$\mathrm{Spin}(4,\mathbb{C}) = SL(2)_+ \times SL(2)_- \qquad SL(2)_+ \xrightarrow[\sim]{\rho} G_R = SL(2,\mathbb{C}) \curvearrowright W \text{ fund. repr.}$$

$\rightsquigarrow$ $\mathrm{Id} \times \rho : \mathrm{Spin}(4) \subseteq \mathrm{Spin}(4) \times G_R$
W, W^* both are copies of S_+

twisted SUSY alg: $\mathbb{C}^4 + \Pi(\underbrace{S_+ \otimes S_+}_{\mathbb{C} + \Lambda^2_+ V} + S_- \otimes S_+)$

$\exists!$ Spin(4)-inv. odd element $Q \in \mathbb{C}$

Claim: $(1,0)$-theory in $6d \xrightarrow[\text{red.}]{\text{dim.}}$ pure $N=2$ theory in $4d$

$6d$: $A \in \Omega^1(\mathbb{R}^6) \longmapsto A_{4d}, \varphi_1, \varphi_2$

$\psi \in C^\infty(\mathbb{R}^6, S \otimes W)$ $\quad \psi_{4d} \in C^\infty(\mathbb{R}^4, S^+_{4d} \otimes S^+_{2d} \otimes W + S^-_{4d} \otimes S^-_{2d} \otimes W)$

$W = \mathbb{C}^2 \curvearrowleft G_R = SL(2)$

$S^+_{6d} = S^+_{4d} \otimes S^+_{2d} + S^-_{4d} \otimes S^-_{2d}$

__Twist__ Forget about $SO(2)$

$\rho : Spin(4) \longrightarrow SL(2)_R$ as before

If we do this, under new $Spin(4)$-action,

$S^{\pm}_{2d}$ scalars,

W, S^+_{4d}

Now, our fields are A^{4d}, $\underbrace{\varphi_1, \varphi_2}_{\text{scalars}}$,

$\psi_{4d} \in C^\infty(\mathbb{R}^4, \underbrace{S^+ \otimes S^+}_{\Lambda^2 S^+ + Sym^2 S^+} + \underbrace{S^+ \otimes S^-}_{\text{vector rep.}})$

$= \mathbb{C} + \Lambda^2_+(\text{vector rep.})$

So $\psi_{4d} \rightsquigarrow$ vector, scalar, self-dual 2-form. (still fermionic).

__More general Twisting__

SUSY theory w/ R-symmetry G_R

$\rightsquigarrow$ can be defined on $G_R \longrightarrow P \longrightarrow (M^d, g)$ spin w/ connection.

e.g. Choose $\rho : Spin(d) \longrightarrow G_R$,

$P := Fr \times_{Spin(d) \circlearrowleft \rho} G_R$ (Witten's twisting)

__More generally__, if $Spin(d) \times G_R \geqslant H$ preserves Q w/ Spinor Q s.t. $Q^2 = 0$

Then given $G_R \longrightarrow P \longrightarrow (M^d, g)$ w/

$Spin(d) \times G_R \longrightarrow Fr \times P \longrightarrow M$ reduced to H

$\rightsquigarrow$ SUSY theory w/ supercharge Q, $Q^2 = 0$.

Claim: 4d $\mathcal{N}=1$ pure gauge theory $\xrightarrow{\text{Holomorphic twist}}$ Holomorphic BF theory

$$\Omega^{0,1}(X,\mathfrak{g}) \times \Omega^{2,0}(X,\mathfrak{g}) \xrightarrow{S} \mathbb{C}$$

$$S(A,B) = \int B \wedge F^{0,2}(A)$$

Rewrite $\mathcal{N}=1$ gauge theory

w/ $B \in \Omega^2_+(X,\mathfrak{g}) \quad (\supset \Omega^{0,2}(X,\mathfrak{g}))$

$$S(A,B,\psi) = \int B \wedge F(A)_+ - \int B \wedge B + \int \psi \not{\partial}_A \psi$$

"Change of coordinates," $B \mapsto B + \frac{1}{2}F(A)_+$,

$$\rightsquigarrow S = \frac{1}{4}\int F(A)_+ \wedge F(A)_+ - \int B \wedge B + \int \psi \not{\partial}_A \psi.$$

$\int B \wedge B$ term EOM : $B=0 \Rightarrow$ remove B.

Claim. Action of $Q \in S_+$ ($\Rightarrow$ CY2) is

$$(A,B,\psi_+,\psi_-) \mapsto (A+\varepsilon\Gamma(Q,\psi_-),\ B,\ \underbrace{\psi_+ + \varepsilon B\cdot Q}_{\text{replace } F_A \text{ by } B},\ \psi_-)$$

$\Rightarrow$ No derivatives!

Fields	bosonic			fermionic	
A	Ω^{10}		Ω^{01}	Ω^{01}	ψ_-
B	Ω^{20}	$\mathbb{C}\omega$	$\Omega^{0,2}$	$\mathbb{C}\omega$ Ω^{02}	ψ_+

SUSY transf ($\Rightarrow$ cancel). $\not\exists$ derivative ($\because$ CY).

susy: $\mathbb{C}\omega \to \mathbb{C}\omega$; susy: $\Omega^{0,2} \to \Omega^{02}$; SUSY transf: $\Omega^{01} \to \Omega^{01}$

Left with $(A^{1,0}, B^{2,0}) \rightsquigarrow S = \int B^{2,0} \wedge F^{0,2}(A)$

4 dimensions

$\int |F|^2 + \int \psi \not{D}_A \psi$ introduced an auxiliary field $B \in \Omega^2_+ \otimes \mathfrak{g}$

Equivalent action $\int F \wedge B + B \wedge B + \int \psi \not{D}_A \psi$

<u>Key point</u>: $\int F_+ \wedge F_+ = \int F \wedge * F + c \underbrace{\int F \wedge F}_{\text{top., does not matter}}$

If X is Kähler of $\dim_{\mathbb{C}} X = d$, let

$\Omega^2_+(X) \subseteq \Omega^2(X)$ be the subspace where $-\int \alpha \wedge \bar{\alpha}$ is pos. definite.

On $\Omega^2_{\pm}(X)$, $\int \alpha \wedge \bar{\alpha} \wedge \omega^{d-2} = \mp \int \alpha \wedge * \bar{\alpha}$

$$\Omega^2_+(X) = \Omega^{2,0}(X) + \omega\, \Omega^{0,0}(X) + \Omega^{0,2}(X)$$

$$\Omega^2_-(X) = \omega^\perp \subseteq \Omega^{1,1}(X)$$

For X^4 Kähler, $S_{ym} \sim \int F(A) \wedge B + B^2$, $B \in \Omega^2_+(X, \mathfrak{g})$

d	supercharges	
$d = 3$	2	$\nexists$ twist
4	4	$SU(2)$-inv. holo. twist
6	8	$SU(3)$-inv. holo. twist
10	16	$SU(5)$-inv. holo. twist

§ 6d, $\mathcal{N}=(1,0)$ pure Gauge theory/CY3

$\xrightarrow{\text{twist}}$ Holomorphic BF theory

$$\Omega^{0,1}(X,\mathfrak{g})\times\Omega^{3,1}(X,\mathfrak{g}) \xrightarrow{S} \mathbb{C}$$

$$S(A,B) = \int B\wedge F^{0,2}(A)$$

Extra gauge symmetry $\chi\in\Omega^{3,0}(X,\mathfrak{g})$, $B\mapsto B+\bar\partial_A\chi$.

$\mathcal{N}=(1,0)$ in $6d$:

Bosons			Fermions	
$A\in$	Ω^{10}	Ω^{01}	$\psi_1\in S_+ \overset{\because CY3}{=\!=}$	$\Omega^{00}+\Omega^{02}$
$B\in\Omega^{20}$	$\omega\Omega^{00}$	Ω^{02}	$\psi_2\in S_- =\!=$	$\Omega^{30}+\Omega^{10}$

$(\mathbb{R}^6\otimes_{\mathbb{R}}\mathbb{C} = \mathbb{C}^3+\overline{\mathbb{C}}^3)$

$Cl(\mathbb{R}^6)\otimes_{\mathbb{R}}\mathbb{C} = \mathbb{C}[e_i,f_j]$, $e_i\in\mathbb{C}^3$ $e_j\in\overline{\mathbb{C}}^3$. $\{f_j,e_j\}=\delta_{ij}$

irrep ↓

$\mathbb{C}[e_i]=\Lambda^\bullet\mathbb{C}^3$, f_j act by $\frac{\partial}{\partial e_j}$

Boson	A		Ω^{10}	Ω^{01}	
	B	Ω^{20}	$\omega\Omega^{00}$	Ω^{02}	
		↑ Id	↓ Id	↓ Id	SUSY, ∄ derivative in act? } ⇒ cancel.
Fermion	ψ_1		Ω^{00}	Ω^{02}	
	ψ_2	Ω^{10}	Ω^{30}		

Remaining fields: $A^{01}\in\Omega^{01}$, $B^{20}\in\Omega^{20}\xrightarrow[\sim]{\wedge\omega}\Omega^{31}$

$\psi_2^{30}\in\Omega^{30}$

Action is $\int B\wedge F^{0,2}(A)$

ψ^{30} is ghost for gauge symmetry $B\to B+\bar\partial_A\psi^{3,0}$

($\xrightarrow{\text{use supercharge}}$ SUSY transformation)

Twists of $\mathcal{N}=2$ theories

4d pure $\mathcal{N}=2$ theory is reduction of 6d (1,0) theory.

Reduce from $\mathbb{C}^3 \longrightarrow \mathbb{C}^2$

$$A^{\mathbb{C}^3}_{\bar z_1, \bar z_2} \rightsquigarrow (0,1)\text{-form on } \mathbb{C}^2 \quad A^{(0,1)}$$
$$A^{\mathbb{C}^3}_{\bar z_3} \rightsquigarrow \text{scalar on } \mathbb{C}^2 \quad \varphi^{(0,0)}$$
$$B_{\bar z_1 \bar z_2} \rightsquigarrow (2,1) \text{ on } \mathbb{C}^2 \quad \tilde A^{(2,1)}$$
$$B_{\bar z_3} \rightsquigarrow \text{scalar on } \mathbb{C}^2 \quad \tilde\varphi^{(2,2)}$$

$$S = \underbrace{\int \tilde\varphi F^{02}(A)}_{\text{pure } \mathcal{N}=1} + \underbrace{\int \varphi \bar\partial_A \tilde A}_{\text{adjoint matter for } \mathcal{N}=1}$$

$\mathcal{N}=2$ gauge theory/CY2 = holom. BF theory w/ $\mathfrak{g} + \overbrace{\mathfrak{g}^*}^{\text{matter}}$.

- Couple with matter fields.

Given $\mathfrak{g} \curvearrowright V$,

$d=4$

holom. twist of $\mathcal{N}=1$ gauge theory w/ matter V

$\equiv$ holom. BF theory w/ Lie alg. $\mathfrak{g} \oplus V$.

Fields: $A + B \in \Omega^{01+00}(X, \mathfrak{g})$

$\chi \in \Omega^{01}(X, V^*) \qquad \varphi \in \Omega^{00}(X, V)$

Action: $S = \int (B \wedge F^{02}(A) + \chi \wedge \bar\partial_A \varphi) \wedge \Omega^{20}$

Gauge $\quad \chi \mapsto \chi + \bar\partial_A \eta \quad$ w/ $\eta \in \Omega^{00}(X, V^*)$

Eg. $\mathfrak{g} = 0$ & $V = \mathbb{C}$ (i.e. free Chiral).

$\int \varphi \bar\partial \chi \qquad \varphi \in \Omega^{00}(\mathbb{C}^2), \ \chi \in \Omega^{01}(\mathbb{C}^2)$

$\chi \mapsto \chi + \bar\partial \eta \quad$ w/ $\eta \in \Omega^{20}(\mathbb{C}^2)$

§ SUSY index

$\mathcal{N}=1,\ d=4$ SUSY index ($G=1 \curvearrowright V=\mathbb{C}$)

$$Z(S^3\times S^1) = \mathrm{Tr}_{\{\text{local operators}\}}\, q_1^{z_1\frac{\partial}{\partial z_1}}\, q_2^{z_2\frac{\partial}{\partial z_2}} \quad \leftarrow \text{scaling in } z_1, z_2$$

$$S^3\times S^1 = (\mathbb{C}^2\setminus 0)/\mathbb{Z} \qquad (z_1,z_2)\sim(2z_1,3z_2)$$

vector fields: $z_j\frac{\partial}{\partial z_j}$'s

Local operators:

Bosonic $\partial_{z_1}^k\partial_{z_2}^\ell\varphi(0)$. (charge (k,ℓ), i.e. weight for $U(1)^2\curvearrowright\mathbb{C}^2$)

∄ operators using χ (∵ $\bar\partial\chi=0 \overset{\text{loc op.}}{\Rightarrow} \chi=\bar\partial\eta$ gauge)

Build operators using η, $\partial_{z_1}^k\partial_{z_2}^\ell\eta(0)$, charge $(k+1,\ell+1)$

$$\left\{\begin{matrix}\text{local}\\ \text{operators}\end{matrix}\right\} \cong S^\bullet\underbrace{\mathbb{C}[\partial_{z_1},\partial_{z_2}]}_{\varphi}\otimes\Lambda^\bullet(\underbrace{\partial_{z_1}\partial_{z_2}\mathbb{C}[\partial_{z_1},\partial_{z_2}]}_{\eta})$$

$$\Rightarrow \text{Character} = \prod_{k,\ell\geq 0}\frac{1-q_1^{k+1}q_2^{\ell+1}u^{-1}}{1-q_1^k q_2^\ell u} \quad \checkmark.$$

u: $U(1)$-action changing scalar.

• 'Twist' — Given a theory on $\mathbb{C}^n$

Twist [Use $U(n)\xrightarrow{\det}U(1)\longrightarrow G_R$
$U(1)$ rotate the supercharge we use.]

$\Longrightarrow$ $U(n)$-invariant on $\mathbb{C}^n$

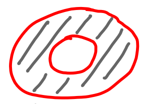
SUSY index $= Z$ on $(\mathbb{C}^n\setminus 0)/\mathbb{Z}\simeq S^{2n-1}\times S^1$

w/ $(z_1,\dots,z_q)\sim(q_1z_1,\dots,q_nz_n)$, $\quad |q_i|<1$

Fundamental domain is between spheres $\sum|z_i|^2=1$ & $\sum|q_iz_i|^2=1$.

Assume we have a CFT.

$\mathcal{H}=Z(S^{2n-1})$ = Space of local operators.

The map induced by the cobordism  is given by applying $\mathrm{Diag}(q_1,\dots,q_n)$ to local operators.

$$Z(S^1\times S^{2n-1}) = \mathrm{Tr}_{\mathcal{H}}\left(\mathrm{Diag}(q_1,\dots,q_n)\right).$$

§ Chern-Simons theory

$A \in \Omega^1(M^3, \mathfrak{g})$ fields

action $\int CS(A)$, $CS(A) = \frac{1}{2}\langle A, dA\rangle + \frac{1}{6}\langle A, [A,A]\rangle$

EOM : $F_A = 0$ flat connections.

$\{\text{BV fields}\} = \Omega^*(M, \mathfrak{g})[1]$

-1	$\Omega^0(M,\mathfrak{g})$	ghosts	1	$\Omega^2(M,\mathfrak{g})$	anti-fields
0	$\Omega^1(M,\mathfrak{g})$	fields	2	$\Omega^3(M,\mathfrak{g})$	anti-ghosts

BV action functional : $S(A) = \int CS(A)$

w/ $A = A_0 + A_1 + A_2 + A_3$

i.e. $S(A) = \int CS(A_1) + \int dA_0 A_2 + \frac{1}{2}\int [A_0, A_0] A_3 + \cdots\cdots$

More generally, given (i) Lie alg. + invariant pairing.

(ii) $\mathcal{A}$ = commutative DGA (eg. $\Omega^\bullet(M)$, $\Omega^{0,\bullet}(X)$)

(iii) odd map $\int : \mathcal{A} \to \mathbb{C}$ s.t. $\int d\alpha = 0$.

$\rightsquigarrow$ action functional $S : \mathcal{A} \otimes \mathfrak{g}[1] \longrightarrow \mathbb{C}$

$$S(\alpha) = \int \tfrac{1}{2}\langle \alpha, d\alpha\rangle + \tfrac{1}{6}\langle \alpha, [\alpha,\alpha]\rangle.$$

Eg. $\mathcal{A} = \Omega_c^{0,*}(\mathbb{C}^{n=\text{odd}})$ w/ $\int \alpha = \int_{\mathbb{C}^n} \alpha \wedge dz_1 \wedge \cdots \wedge dz_n$.

Field theory $n=3$ is holomorphic Chern-Simons

$n=3$, call holomorphic Chern-Simons (hCS)

Eg $\mathcal{A} = \underset{dx_i}{\Omega_c^*(\mathbb{R}^k)} \otimes \underset{d\bar{z}_j}{\Omega_c^{0,*}(\mathbb{C}^\ell)}$, $d_{\mathcal{A}} = \sum dx_i \frac{\partial}{\partial x_i} + \sum d\bar{z}_j \frac{\partial}{\partial \bar{z}_j}$

w/ $k+\ell$ odd

$$\int \alpha := \int_{\mathbb{R}^k \times \mathbb{C}^\ell} \alpha \wedge dz_1 \wedge \cdots \wedge dz_\ell.$$

- $k=2$, $\ell=1$ $\rightsquigarrow$ Yangian / integrable models.

Eg. (w/ superdirections) $\mathbb{R}^k \times \mathbb{C}^\ell \times \mathbb{C}^{0|m}$ w/ $k+\ell+m$ odd

$$\mathcal{A} = C_c^\infty(\mathbb{R}^k \times \mathbb{C}^\ell)[dx_i, d\bar{z}_j, \varepsilon_r] \text{ w/ } d_{\mathcal{A}} = d_{dR}^{\mathbb{R}^k} + \bar{\partial}^{\mathbb{C}^\ell}$$

$$\int \alpha = \int_{\mathbb{R}^k \times \mathbb{C}^\ell \times \mathbb{C}^{0|m}} \alpha \, dz_1 \cdots dz_\ell \, d\varepsilon_1 \cdots d\varepsilon_m$$

($\int_{\mathbb{C}^{0|m}}$ picks out coeff of $\varepsilon_1 \cdots \varepsilon_m$)

Fields $\Omega^\bullet(\mathbb{R}^k) \otimes \Omega^{0,\bullet}(\mathbb{C}^m)[\varepsilon_1, \ldots, \varepsilon_n] \otimes \mathfrak{gl}_N[1] \ni \alpha$

EOM = bundles on $\mathbb{R}^k \times \mathbb{C}^{m|n}$, flat on $\mathbb{R}^k$, hol. on $\mathbb{C}^{m|n}$

<u>Thm</u> (Baulieu)

10d, $\mathcal{N}=1$ gauge th., twist by $SU(5)$-inv $Q \in S$

= hCS on $\mathbb{C}^5$ (or CY5)

$$\Omega^{0,*}(\mathbb{C}^5) \otimes \mathfrak{g}[1] = \{\text{fields}\}$$

Dimension reduction : replace $\mathbb{C}^k$ by $\mathbb{C}^{0|k}$.

$\Rightarrow$ max. SUSY twisted theory in $d = 2k$

$$\underset{\dim_{\mathbb{R}}=16}{S^+_{10d}} = \underbrace{S^+_{2k} \otimes S^+_{10-2k}}_{\ni Q} + S^-_{2k} \otimes S^-_{10-2k}$$

Q : $SU(5)$-inv. $\Longrightarrow$ $SU(k) \times SU(5-k)$-inv.

Assume $Q = \psi \otimes e$ decomposable.

(otherwise, need to preserve ≥ 2 cpx. str.).

Eg. 4d, $\mathcal{N}=4$ YM

Supersymmetries are $S^+_{4d} \otimes W + S^-_{4d} \otimes W^*$

$W = \mathbb{C}\langle e_1, e_2, e_3, e_4 \rangle$; $\quad \psi \otimes e_1 =: Q$

Q - twist $\Rightarrow$ hCS on $\mathbb{C}^{2|3} \subset \mathbb{P}^{2|3}$ $\leftarrow$ $psu(3|3)$

$$\{\text{Fields}\} = \Omega^{0,*}(\mathbb{C}^2)[\varepsilon_i] \otimes \mathfrak{g}[1]$$

$(\mathcal{N}=4) = \mathcal{N} = 1 + 3$ Chiral fields in adjoint ($1 \leftarrow e_1$)

(can be viewed as $\mathcal{N}=1$ th. w/ 3 chiral multiplets).

$A \in \Omega^{0,1}(\mathbb{C}^2) \otimes \mathfrak{g}$ — Gauge field

$B \in \varepsilon_1 \varepsilon_2 \varepsilon_3 \Omega^{0,0}(\mathbb{C}^2) \otimes \mathfrak{g}$ — B part of curvature

Action for these 2 fields is $\int B\, F(A)\, dz_1 dz_2$

$\varepsilon_i \Omega^{0,0}(\mathbb{C}^2) \otimes \mathfrak{g}$ — 3 chiral fields

$\varepsilon_i \Omega^{0,1}(\mathbb{C}^2) \otimes \mathfrak{g}$ — fermions in $\mathcal{N}=1$ Chiral multiplet

$\varepsilon_i \Omega^{0,2}(\mathbb{C}^2) \otimes \mathfrak{g}$ — auxillary fields

$\varepsilon_i \varepsilon_j \Omega^{0,*}(\mathbb{C}^2) \otimes \mathfrak{g}$ — anti-fields

<u>Proposition</u>. The vector fields $\frac{\partial}{\partial z_i}$ (bosonic), $\varepsilon_i \frac{\partial}{\partial z_j}$, $\frac{\partial}{\partial \varepsilon_i}$ acting on $\mathbb{C}^{2|3}$ (and so on the theory).

$\varepsilon_i \frac{\partial}{\partial z_j}$ (6), $\frac{\partial}{\partial \varepsilon_i}$ (3) are the remaining supersymmetries.

$$\left[\frac{\partial}{\partial \varepsilon_i}, \varepsilon_j \frac{\partial}{\partial z_k}\right] = \delta_{ij} \frac{\partial}{\partial z_k} \quad \text{(bosonic translations)}.$$

Physical SUSY

$S^+ \otimes \mathbb{C}^4 + S^- \otimes \mathbb{C}^4$

$e_1, \dots, e_4$; $e_1^*, \dots, e_4^*$

$Q = \psi \otimes e_1$

Symmetries that survive twisting commute w/ Q.

All of $S^+ \otimes \mathbb{C}^4$ commutes w/ Q $\quad (S^+ \otimes S^+ \simeq (\Lambda^0 V + \Lambda^2_+ V)_{\mathbb{C}})$

$[Q, -] : S^- \otimes e_1^* \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C} \quad (S^+ \otimes S^- \simeq V_{\mathbb{C}})$

image spanned by $\frac{\partial}{\partial z_j}$, Q-exact.

$S^+ \otimes \mathbb{C}^4 + S^- \otimes \mathbb{C}^3$ commute w/ Q

$[Q, so(4) + sl(4)_R]$ will be zero after twisting.

This space is $\psi \otimes \mathbb{C}^4 + S^+ \otimes e_1$.

In Q-cohomology gives $\psi' \otimes e_2, e_3, e_4$, ($\psi' \in S_+$ linear indep. from ψ)

$3+6$ SUSY survive: $\frac{\partial}{\partial \varepsilon_i}$ $i = 1, 2, 3$ (3), $\varepsilon_i \frac{\partial}{\partial z_j}$ (6)

Theorem. Maximal SUSY theory in $2k < 10$ dim. is hCS on $\mathbb{C}^{k|5-k}$. The residual SUSYs are precisely the vector fields $\varepsilon_i \frac{\partial}{\partial z_j}$ ($k(5-k)$) and $\frac{\partial}{\partial \varepsilon_i}$ ($(5-k)$).

Theorem. In $d = 2k+1 < 10$,

max. SUSY gauge theory

$\equiv$ twisted CS on $\mathbb{R} \times \mathbb{C}^{k|4-k}$

Residual SUSY = $\varepsilon_i \frac{\partial}{\partial z_j}$'s + $\frac{\partial}{\partial \varepsilon_i}$'s.

(see non-linear example below).

§ Twisted max. SUSY theory / non-linear base, $X \times \mathbb{R}$

$$\mathbb{C} \to L \to X^{2c} \qquad \tilde{X} \text{ odd rk 2 bdl.} / X$$

$$\sim \text{CS on } \mathbb{R} \times (\underbrace{\Pi(L \oplus K^{-1} \otimes L^{-1})}_{\tilde{X}}) \qquad \left(\text{nonlinear of } \mathbb{R} \times \mathbb{C}^{2|2}\right)$$

super CY

EOM of the theory

$= G$-bundles on $\mathbb{R} \times \tilde{X}$ (holo. in $\tilde{X}$, flat in $\mathbb{R}$)

$= \mathrm{Bun}_G(\tilde{X})$ ⟵ super-symplectic

$=$ phase space $= T^* \mathrm{Bun}_G(\Pi L)$.

$$\begin{array}{ccc} \mathrm{Bun}_G(\Pi L) & \supseteq & H^0(X, L \otimes \mathrm{ad} E). \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G(X) & \ni & [E] \end{array}$$

When $L = \mathcal{O}_X$,

$$\begin{aligned} \mathrm{Bun}_G(\Pi L) &= \mathrm{Bun}_G(\mathbb{C}^{0|1} \times X) \\ &= \mathrm{Maps}(\mathbb{C}^{0|1}, \mathrm{Bun}_G(X)) \\ &= \Pi T\, \mathrm{Bun}_G(X) \end{aligned}$$

$$\begin{aligned} \text{Hilbert space} &= \Omega^{0,*}(\mathrm{Bun}_G(\Pi L)) \\ &= \Omega^{0,*}(\Pi T\, \mathrm{Bun}_G(X)) \\ &= \bigoplus \Omega^{0,*}(\mathrm{Bun}_G X, \Lambda^\bullet T_{\mathrm{Bun}_G(X)}) \end{aligned}$$

$$\begin{aligned} Z(S^1 \times X) &= \mathrm{Tr}_{\mathcal{H}}\, 1 \\ &= \chi(\mathrm{Bun}_G(X)) \end{aligned}$$

✓✓ Vafa-Witten.

§ 7d Gauge theory $\rightsquigarrow$ K-theory DT-inv.

- twist 7d Gauge theory/$\mathbb{R}^7$ $=$ CS/$\mathbb{C}^{3|1} \times \mathbb{R}$

$$\Omega^{0,*}(\mathbb{C}^3) \otimes \Omega^*(\mathbb{R})[\varepsilon] \otimes \mathfrak{g}[1] \longrightarrow \mathbb{C}$$

$$\varphi \longmapsto \int CS(\varphi)\, dz_1 dz_2 dz_3\, d\varepsilon$$

- / $X_{CY3} \times \mathbb{R}$

Phase space $= \{$ sol$^{\underline{n}}$ of EOM/ $X \times (-\varepsilon, \varepsilon)\}$ always sympl.

$$\Omega^{0,*}(X)[\varepsilon] \otimes \mathfrak{g}[1]$$ ← indep. of t & dt. (∵ CS involve only 1st derivative)

deg. ε: -1, $[1]$: $+1$

deg 0 part $\rightsquigarrow$ $A \in \Omega^{0,1}(X) \otimes \mathfrak{g}$, $B \in \Omega^{0,2}(X) \otimes \mathfrak{g}$

EOM: $F_A^{0,2} = 0 = \bar{\partial}_A B$ mod. gauges

7d gauge/$X \times \mathbb{R}$ $\xrightarrow[\text{on } X]{\text{compactify}}$ Q.M. on $T^* Bun_G(X)$

$\Rightarrow$ Hilbert space $H^{\bullet}_{\bar{\partial}}(Bun_G(X), \mathcal{K}^{\otimes 1/2})$ Geometric quantization

$$Z(X \times S^1) = \chi(\text{——}''\text{——})$$

i.e. K-theoretic D.T.-inv.

§ Theory w/ 8 supercharges

$\Rightarrow$ matter V w/ $\mathfrak{g} \longrightarrow sp(V, \omega)$

Theory w/ 16 supercharges

$\Rightarrow$ $V = T^*\mathfrak{g}$

Matter $\mathfrak{g} \to sp(V, \omega_V)$ ($\rightsquigarrow$ quadratic $V \xrightarrow[\text{moment}]{\mu} \mathfrak{g}^*$)

$\rightsquigarrow$ graded Lie alg. $\mathfrak{g}_V := \mathfrak{g} + V + \mathfrak{g}^*$ (deg: 0, 1, 2)

$[v_1, v_2] = \partial_{v_1}\partial_{v_2}\mu$ $\qquad$ (x, v, x^*)

- $\alpha \in \mu^{-1}(0) \subset V$

$\Leftrightarrow$ $\alpha \in \mathfrak{g}_V$, deg 1, s.t. $[\alpha, \alpha] = 0$ (MC eqt)

- $\mathfrak{g}_V^0 \curvearrowright \{\text{MC sol}^{\text{n}}\}$ via $\alpha \mapsto \alpha + \varepsilon[x, \alpha]$

- $V/\!/\mathfrak{g} = \mu^{-1}(0)/\mathfrak{g}$ = moduli of MC sol$^{\text{n}}$.

<u>Theorem.</u> Twisted $6d$ $\mathcal{N} = (1,0)$ w/ matter V

$= S : \Omega^{0,*}(\mathbb{C}^3) \otimes \mathfrak{g}_V[1] \to \mathbb{C}$

$$S(\varphi) = \int_{\mathbb{C}^3} \left(\frac{1}{2}\langle \varphi, d\varphi \rangle + \frac{1}{6}\langle \varphi, [\varphi, \varphi] \rangle \right) dz_1 dz_2 dz_3$$

(Here $\langle \ \rangle$: $\mathfrak{g} + V + \mathfrak{g}^*$, with (ω_V) on V and natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$)

— Holom. Rozansky-Witten theory.

- On CY3 X, EOM = ?

$\mathfrak{g}$-part $\Rightarrow$ $G \to P \to X$ (holo. G-bdl)

V-part $\Rightarrow$ $\varphi \in \Omega^{0,0}(X, P \times_G V)$ s.t. $\bar{\partial}_A \varphi = 0$

$\mathfrak{g}^*$-part $\Rightarrow$ $\mu(\varphi) = 0 \in \Omega^{0,0}(X, \mathfrak{g})$

- If $\varphi \in U \overset{\text{open}}{\subseteq} V$ w/ $G \overset{\text{free}}{\curvearrowright} U$

then $\{\text{such EOM sol}^{\text{n}}\} = \{X \xrightarrow{\text{holo.}} U/\!/G\}$

" $V/\!/G$: Higgs branch "

- Reduction to 5d ⇝ K-th. Donaldson inv, ...

- Reduction to 3d ⇝ $\mathbb{C}^{1|1}\times\mathbb{R}$ and $\mathfrak{g}_V$

$$\Omega^{0,*}(\mathbb{C})\otimes\Omega^{*}(\mathbb{R})[\varepsilon]\otimes\mathfrak{g}_V[1]=\{\text{fields}\}$$

⇝ twisted 3d $\mathcal{N}=4$ gauge theory

SUSY: $\varepsilon\partial_z$ $\overset{\text{twist}}{\rightsquigarrow}$ RW twist (B)

∂_ε ⇝ twised RW (A)

(B) ↔ (A): 3d mirror symmetry

$$[\partial_\varepsilon,\varepsilon\partial_z]=\partial_z$$

(degree) $\geq 1 \Rightarrow$ BRST-exact.

Eg. $\mathfrak{g}=0$, i.e. $\varphi\in\Omega^{0,\bullet}(\mathbb{C})\otimes\Omega^{\bullet}(\mathbb{R})\otimes V[\varepsilon]$

write $\varphi=\varphi_0+\varepsilon\varphi_1$

EOM: $(\bar\partial_z+d_{dR})\varphi=0$

operators ~ functions of

$\partial_z^k\varphi_0(0)$ (bosonic), $\partial_z^k\varphi_1(0)$ (fermionic)

$$\partial_z^k\varphi_0(0)\overset{\partial_\varepsilon}{\longmapsto}\partial_z^k\varphi_1(0),\quad \partial_z^k\varphi_1(0)\overset{\varepsilon\partial_z}{\longmapsto}\partial_z^{k+1}\varphi_0(0)$$

$$H^\bullet(\partial_\varepsilon)=0,\quad H^\bullet(\varepsilon\partial_z)=\mathbb{C}\langle\varphi_0(0)\rangle$$

$\varepsilon\partial_z$ is RW twist. ⇒ {local operators} $=S^\bullet V$

In general $\mathfrak{g}\neq0$, {loc. operator} = {holo. fu. on $V/\!/G$}